Why Quarks are Different from Leptons – An Explanation by a Fermionic Substructure of Leptons and Quarks

H. Stumpf

Institute of Theoretical Physics, University Tuebingen, Germany

Reprint requests to Prof. H. S.; E-mail: herold@tat.physik.uni-tuebingen.de

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To explain the difference between leptons and quarks, it is assumed that electroweak gauge bosons, leptons and quarks are composites of elementary fermionic constituents denoted by partons (not to be identified with quarks) or subfermions, respectively. The dynamical law of these constituents is assumed to be given by a relativistically invariant nonlinear spinor field theory with local interaction, canonical quantization, selfregularization and probability interpretation. According to the general requirements of field operator algebraic theory, this model is formulated in algebraic Schroedinger representation referred to generating functionals in functional state spaces. The derivation of the corresponding effective dynamics for the composite particles is studied by the construction of a map between the spinor field state functionals and the state functionals of the effective theory for gauge bosons, leptons and quarks. A closer examination of this map shows that it is then and then only selfconsistent if certain boundary conditions are satisfied. The latter enforce in the case of electroweak symmetry breaking the difference between lepton and quark states. This difference can be analytically expressed as conditions to be imposed on the wave functions of these composite particles and leads ultimately to the introduction and interpretation of color for quarks, i.e., the characteristic of their strong interaction.

Key words: Substructure of Quarks and Leptons; Effective Dynamics; Difference of Leptons and Quarks.

1. Introduction

In the standard model of elementary particle physics leptons and quarks are assumed to constitute the fermionic part of its basic particle set, i.e. in this model leptons and quarks are considered to be the ultimate fermionic constituents of matter without substructure.

In spite of the enormous success of this model with respect to the experimental verification, it raises a lot of theoretically and experimentally unsolved questions, see [1], Sect. 3.10, [2], Sect. 5.7, [3], Sect. 12, and these have led to further alternative model building since several decades, see [1].

One of these unsolved problems is the existence of lepton and quark families, or more simply: Why quarks are different from leptons and why do they exist in generations?

If one tries to answer these special questions, one is confronted with the basic problem by which kind of model the standard model should be replaced, because within the standard model itself one cannot find any reason for the difference of leptons and quarks and their replication.

At present the general tendency prevails to embed the standard model into theories with grand unification, supersymmetry, superstrings, etc., see [1]. But to solve the above problem we prefer the compositeness hypothesis as a possible approach of model building.

While the former model building leads to an enlargement of symmetries and (or) dimensions, etc., the latter approach should lead to a simplification of the basic particle set and of the interactions. Thus from an economic point of view the compositeness hypothesis is very attractive. Nevertheless, before starting a theoretical discussion one should have a look on experiments. Does there exist any experimental hint in favour of this hypothesis?

In high energy physics deep inelastic scattering processes of photons by photons and similarly of quarks by photons seem to provide signals of compositeness of the photons themselves, [4 – 6], as well as of the quarks, [7 – 9]. In addition an unexpected confirmation of compositeness comes from the study of superfluids with an enormous variety of experimental facts and analogies to high energy physics and even cosmology, [10]. We consider these facts as a justification of theoretical research in compositeness models.

In the past decades such a model was developed which we use as a suitable candidate to treat quantitatively the substructure of elementary particles. This model is based on a relativistically invariant nonlinear spinor field theory with local interaction, canonical quantization, selfregularization and probability interpretation. It can be considered as the quantum field theoretic generalization of de Broglie's theory of fusion, [11], and as a mathematical realization and physical modification of Heisenberg's approach, [12], and is expounded in [13, 14].

Its basic particle set consists of partons (not to be identified with quarks) or subfermions, respectively, which are assumed to be the elements of the substructure of the elementary particles of the standard model. In this model the formation of additional new partonic bound states is not excluded, where the latter can act as additional "elementary particles" in the corresponding effective theory replacing the standard model.

In analyzing the problem why quarks are different from leptons, we refer to this model. In particular, in its simplest version the electroweak gauge bosons are assumed to be two-parton bound states, while the fermion families are assumed to arise from three parton bound states. The latter assumptions are made in several compositeness models, [15]. Among other authors, the three fermion substructure of leptons and quarks was proposed by Harari [16] and Shupe [17]. But apart from this assumption our model has nothing in common with the Harari-Shupe model or other models cited in [15], because the mathematical formulation and evaluation is different from previous attempts in this field.

In previous attempts, see [15], the difference between leptons and quarks was generally explained by attributing to quarks special algebraic and group theoretical degrees of freedom, whereas in the case under consideration we start with a globally invariant $SU(2) \otimes U(1)$ spinor theory, and the difference between leptons and quarks is generated by the dynamics of the model itself. Being composite particles, the effective dynamics of leptons, quarks and gauge bosons will be studied. In this case the consistence of the mapping from the basic parton theory to the corresponding effective theory imposes subsidiary conditions which enforce the difference between lepton and quark states and hence offers an explanation of the difference of leptons and quarks themselves.

Thus, before discussing the latter problem we first develop the formal proofs for the derivation of the effective dynamics of these composite particles and, subsequently, based on the results of these investigations, we can treat the actual problem in the last section.

2. Algebraic Representation of the Spinor Field

The algebraic representation is the basic formulation of the spinor field model and the starting point for its evaluation. In order to avoid lengthy deductions we give only some basic formulas of this formalism and refer for details to [13, 14, 18]. The corresponding Lagrangian density reads, see [14], Eq. (2.52)

$$\begin{aligned}\mathcal{L}(x) := {} & \sum_{i=1}^{3} \lambda_i^{-1} \bar{\psi}_{A\alpha i}(x)(i\gamma^\mu \partial_\mu - m_i)_{\alpha\beta} \delta_{AB} \psi_{B\beta i}(x) \\ & - \frac{1}{2} g \sum_{h=1}^{2} \delta_{AB} \delta_{CD} v^h_{\alpha\beta} v^h_{\gamma\delta} \sum_{i,j,k,l=1}^{3} \bar{\psi}_{A\alpha i}(x) \psi_{B\beta j}(x) \bar{\psi}_{C\gamma k}(x) \psi_{D\delta l}(x)\end{aligned} \tag{1}$$

with $v^1 := \mathbf{1}$ and $v^2 := i\gamma^5$. The field operators are assumed to be Dirac spinors with index $\alpha = 1, 2, 3, 4$ and additional isospin with index $A = 1, 2$ as well as auxiliary fields with index $i = 1, 2, 3$ for nonperturbative regularization. The algebra of the field operators is defined by the anticommutators

$$[\psi^+_{A\alpha i}(\mathbf{r}, t) \psi_{B\beta j}(\mathbf{r}', t)]_+ = \lambda_i \delta_{ij} \delta_{AB} \delta_{\alpha\beta} \delta(\mathbf{r} - \mathbf{r}') \tag{2}$$

resulting from canonical quantization. All other anticommutators vanish.

To obtain a uniform transformation property with respect to Lorentz transformations, the adjoint spinors are replaced by formally charge conjugated spinors, which are defined by

$$\psi^c_{A\alpha i}(x) = C_{\alpha\beta} \bar{\psi}_{A\beta i}(x), \tag{3}$$

and the index Λ is introduced by

$$\psi_{A\Lambda\alpha i}(x) = \begin{pmatrix} \psi_{A\alpha i}(x); & \Lambda = 1 \\ \psi^c_{A\alpha i}(x); & \Lambda = 2 \end{pmatrix}. \tag{4}$$

Then the set of indices is defined by $Z := (A, \Lambda, \alpha, i)$.

The operator formulation of the theory by means of (1) and (2) cannot be evaluated without further preparation. In order to obtain definite results from this theory a state space is needed in which the dynamical equations can be formulated. This is achieved by the use of the algebraic Schroedinger representation. For a detailed discussion we refer to [13, 14, 18]. Here we present only the results and definitions.

To ensure transparency of the formalism we use the symbolic notation

$$(\psi_{I_1}\dots\psi_{I_n}) := \psi_{Z_1}(\mathbf{r}_1,t)\dots\psi_{Z_n}(\mathbf{r}_n,t) \quad (5)$$

with $I_k := (Z_k, \mathbf{r}_k, t)$. Then in the algebraic Schroedinger representation a state $|a\rangle$ is characterized by the set of matrix elements

$$\tau_n(a) := \langle 0|\mathcal{A}(\psi_{I_1}\dots\psi_{I_n})|a\rangle, \quad n = 1\dots\infty, \quad (6)$$

where $\mathcal{A}$ means antisymmetrization in $I_1 \dots I_n$.

By means of this definition the calculation of an eigenstate $|a\rangle$ is transferred to the calculation of the set of matrix elements (6) which characterize this state. For a compact formulation of this method, generating functionals are introduced, and the set (6) is replaced by the functional state

$$|\mathcal{A}(j;a)\rangle := \sum_{n=1}^{\infty} \frac{i^n}{n!} \sum_{I_1\dots I_n} \tau_n(I_1\dots I_n|a) j_{I_1}\dots j_{I_n}|0\rangle_f, \quad (7)$$

where $j_I := j_Z(\mathbf{r})$ are the generators of a CAR-algebra with corresponding duals $\partial_I := \partial_Z(\mathbf{r})$ which satisfy the anticommutation relations

$$[j_I, \partial_{I'}] = \delta_{ZZ'}\delta(\mathbf{r}-\mathbf{r}'), \quad (8)$$

while all other anticommutators vanish.

With $\partial_I|0\rangle_f = 0$ the basis vectors for the generating functional states can be defined. The latter are not allowed to be confused with creation and annihilation operators of particles in physical state spaces, because generating functionals are formal tools for a compact algebraic representation of the states $|a\rangle$: To each state $|a\rangle$ in the physical state space we associate a functional state $|\mathcal{A}(j;a)\rangle$ in the corresponding functional space. The map is biunique, and the symmetries of the original theory are conserved. Nevertheless this map does not induce any equivalence of the kind indicated above. For details see [13, 14].

In order to find a dynamical equation for the functional states, we apply to the operator products (5) the Heisenberg formula

$$i\frac{\partial}{\partial t}\mathcal{A}(\psi_{I_1}\dots\psi_{I_n}) = [\mathcal{A}(\psi_{I_1}\dots\psi_{I_n}), H]_-, \quad n = 1\dots\infty, \quad (9)$$

where *H* is the Hamiltonian of the system under consideration.

If $|0\rangle$ as well as $|a\rangle$ are assumed to be eigenstates of *H*, then between both states the matrix elements of (9) can be formed, and subsequent evaluation leads to the functional equation, see [13, 14]

$$E_0^a|\mathcal{A}(j;a)\rangle = [K_{I_1I_2}j_{I_1}\partial_{I_2} - W_{I_1I_2I_3I_4}j_{I_1}(\partial_{I_4}\partial_{I_3}\partial_{I_2} + A_{I_4J_1}A_{I_3J_2}j_{J_1}j_{J_2}\partial_{I_2})]|\mathcal{A}(j;a)\rangle \quad (10)$$

with $E_0^a = E_a - E_0$ and summation convention for I, etc.

The symbols which are used in (10) are defined by the following relations:

$$K_{I_1I_2} := iD^0_{I_1I}(D^k\partial_k - m)_{II_2} \quad (11)$$

with

$$D^\mu_{I_1I_2} := i\gamma^\mu_{\alpha_1\alpha_2}\delta_{A_1A_2}\delta_{\Lambda_1\Lambda_2}\delta_{i_1i_2}\delta(\mathbf{r}_1-\mathbf{r}_2), \quad m_{I_1I_2} := m_{i_1}\delta_{\alpha_1\alpha_2}\delta_{A_1A_2}\delta_{\Lambda_1\Lambda_2}\delta_{i_1i_2}\delta(\mathbf{r}_1-\mathbf{r}_2), \quad (12)$$

and

$$W_{I_1I_2I_3I_4} := -iD^0_{I_1I}V_{II_2I_3I_4} \quad (13)$$

with

$$V_{I_1I_2I_3I_4} := \sum_{h=1}^{2} g\lambda_{i_1}B_{i_2i_3i_4}v^h_{\alpha_1\alpha_2}\delta_{A_1A_2}\delta_{\Lambda_1\Lambda_2}(v^hC)_{\alpha_3\alpha_4}\delta_{A_3A_4}\delta_{\Lambda_31}\delta_{\Lambda_42}\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{r}_1-\mathbf{r}_3)\delta(\mathbf{r}_1-\mathbf{r}_4),$$
$$B_{i_2i_3i_4} := 1, \quad i_2,i_3,i_4 = 1,2,3 \quad (14)$$

and the anticommutator matrices

$$A_{I_1I_2} := \lambda_{i_1}(C\gamma^0)_{\alpha_1\alpha_2}\delta_{A_1A_2}\sigma^1_{\Lambda_1\Lambda_2}\delta_{i_1i_2}\delta(\mathbf{r}_1 - \mathbf{r}_2). \tag{15}$$

So far this functional equation holds for any algebraic representation. A special representation can be selected by specifying the corresponding vacuum. This can be achieved by the introduction of normal ordered functionals. Using the summation convention for I they are defined by

$$|\mathcal{F}(j;a)\rangle := \exp[\frac{1}{2}j_{I_1}F_{I_1I_2}j_{I_2}]|\mathcal{A}(j;a)\rangle =: \sum_{n=1}^{\infty}\frac{i^n}{n!}\varphi_n(I_1\dots I_n|a)j_{I_1}\dots j_{I_n}|0\rangle_f, \tag{16}$$

where the two-point function

$$F_{I_1I_2} := \langle 0|\mathcal{A}\{\psi_{Z_1}(\mathbf{r}_1,t)\psi_{Z_2}(\mathbf{r}_2,t)\}|0\rangle \tag{17}$$

contains an information about the groundstate and thus fixes the representation.

The normal ordered functional equation then reads

$$E_0^a|\mathcal{F}(j;a)\rangle = \mathcal{H}_F(j,\partial)|\mathcal{F}(j;a)\rangle \tag{18}$$

with

$$\begin{aligned}\mathcal{H}_F(j,\partial) := {} & j_{I_1}K_{I_1I_2}\partial_{I_2} + W_{I_1I_2I_3I_4}[j_{I_1}\partial_{I_4}\partial_{I_3}\partial_{I_2} - 3F_{I_4K}j_{I_1}j_K\partial_{I_3}\partial_{I_2} + (3F_{I_4K_1}F_{I_3K_2} \\ & + \frac{1}{4}A_{I_4K_1}A_{I_3K_2})j_{I_1}j_{K_1}j_{K_2}\partial_{I_2} - (F_{I_4K_1}F_{I_3K_2} + \frac{1}{4}A_{I_4K_1}A_{I_3K_2})F_{I_2K_3}j_{I_1}j_{K_1}j_{K_2}j_{K_3}].\end{aligned} \tag{19}$$

Equation (19) is the algebraic Schroedinger representation of the spinorfield Lagrangian (1), written in functional space with a fixed algebraic state space. With respect to the physical interpretation of this formalism two comments have to be added:

i) The algebraic Schroedinger representation is the formulation of the Hamilton formalism for quantum fields independently of perturbation theory. Nevertheless, for the justification of this procedure the findings of perturbation theory are indispensible.

From the perturbation theory one learns that during the time interval where mutual interactions between the particles take place, these particles cannot be kept on their mass shell. This means that in order to get nontrivial interactions one is not allowed to enforce the particles on their mass shell all the time.

The formal treatment of perturbation theory starts with the Schroedinger picture, i. e., the Hamilton formalism which explicitly avoids the on shell fixing of particle masses. Hence, if this treatment of the perturbation theory is extended to the case of composite particle interactions, the use of the Hamilton operator leads to the formalism introduced above.

Then one will ask whether in consequence of this procedure relativistic covariance is completely lost. In perturbation theory this is not the case, because the equivalence of the Hamilton formalism and the covariant formulation can be shown. A similar result can be obtained for the composite particle theory in algebraic Schroedinger representation: The corresponding effective theories for composite particle reactions can be covariantly formulated too.

ii) In general, in the literature spinor field models, like the NJL-model in nuclear physics, are considered as effective, low energy theories. If spinor field models are to play a more fundamental role, they need a special preparation to suppress divergencies. In the case under consideration this preparation is expressed by the introduction of auxiliary fields with constants λ_i in (1), (2) and (14), (15), which are designed to generate a nonperturbative intrinsic regularization of the theory.

This regularization is closely connected with the probability interpretation of the theory. As the λ_i are indefinite, an indefinite state space of the auxiliary fields results. Hence these auxiliary fields are unobservable, and a special definition of a corresponding physical state space is needed. This definition is identical with the intrinsic regularization prescription and leads in turn to probability conservation in physical state space. Furthermore it can be shown that the latter property can be transferred to the corresponding effective theories themselves. For details we refer [18].

3. Bosonization in Functional Space

With respect to an application in section 5, we first treat the simple case of collective modes where bound states are represented by fermion pairs. In nuclear physics this treatment has a long history, [19, 20], and in solid state physics it is applied in numerous versions. Following [19] we shortly give the central formulas of the conventional method of approach, which afterwards will be modified to treat the problem under consideration.

Let

$$|n\rangle_f = a^+_{\beta_1} a^+_{\beta_2} \dots a^+_{\beta_{2n-1}} a^+_{\beta_{2n}} |0\rangle_f \tag{20}$$

be an even numbered fermion state in fermionic Fock space with creation operators a^+_β and the corresponding Fock vacuum $|0\rangle_f$, see [19] Eqs. (2.5), (2.6), and let

$$|n\rangle_B = b^+_{\{\beta_1\beta_2} \dots b^+_{\beta_{2n-1}\beta_{2n}\}} |0\rangle_B \tag{21}$$

be the corresponding bosonic Fock space with creation operators $b^+_{\alpha\alpha'}$ and the corresponding Fock vacuum $|0\rangle_B$, where the brackets in (21) mean antisymmetrization $\equiv \mathcal{A}$, see [19], Eqs. (2.1), (2.3). Then the operator

$$\mathcal{U} := \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\alpha_1 \dots \alpha_{2n}} b^+_{\alpha_1\alpha_2} \dots b^+_{\alpha_{2n-1}\alpha_{2n}} |0\rangle_B \; {}_f\langle 0| a_{\alpha_1} a_{\alpha_2} \dots a_{\alpha_{2n-1}} a_{\alpha_{2n}} \tag{22}$$

transforms the fermion state (20) into the boson state (21), see [19], Eq. (2.4), i. e.

$$|n\rangle_B = \mathcal{N}\mathcal{U}|n\rangle_f, \tag{23}$$

where $\mathcal{N}$ is an appropriate normalization factor, and where the exponential in [19], (2.4), is represented by its power series expansion.

If the formulas (21) – (23) are systematically evaluated, then Schroedinger equations in fermionic Fock space can be transformed into Schroedinger equations in bosonic Fock space.

In the case under consideration the algebraic Schroedinger representation of the spinor field replaces the ordinary Schroedinger equation in Fock space, and the corresponding equation is defined by the functional equation (18), (19) in functional space.

A transformation of the latter functional equation to describe fermion pairs leads to a corresponding bosonic functional equation in algebraic Schroedinger representation. For this case, exact mapping theorems were derived in [21, 22] and [23]. In [23] the functional mapping formalism was formulated in close analogy to ordinary bosonization in Fock space.

In the algebraic Schroedinger representation the central formula which defines the map into the boson representation was originally given by the expansion, see [13, 14],

$$\varphi_n(I_1 \dots I_{2n}|a) = \sum_{k_1 \dots k_n} \frac{1}{(2n)!} \varrho(k_1 \dots k_n) C_{k_1}^{\{I_1 I_2} \dots C_{k_n}^{I_{2n-1} I_{2n}\}}, \tag{24}$$

where the wave functions $C_k^{II'}$ are single time functions which result from fully covariant wave functions $\varphi_k(x_1, x_2)$ by formation of the symmetric limit $t_1 \to t$, $t_2 \to t$, and by separation into parts which describe the wave functions of collective variables. So the wave functions $C_k^{II'}$ reflect in their structure their relativistic origin by containing the relativistic deformations, but no genuine energy eigenvalue equation can be derived for them. In addition the state space of the auxiliary fields is indefinite, see Section 2.

From these facts it follows that in general C_k^* is not the dual of C_k. In contrast to the Fock space mapping methods in nuclear physics, it is thus necessary to introduce dual states $R^k_{II'}$, where both types of states are assumed to be antisymmetric functions

$$C_k^{II'} = -C_k^{I'I}; \quad R^k_{II'} = -R^k_{I'I}, \tag{25}$$

and where the relations

$$\sum_{I_1 I_2} C_k^{I_1 I_2} R^{k'}_{I_1 I_2} = \delta_k^{k'}, \tag{26}$$

$$\sum_k C_k^{I_1 I_2} R^k_{I_1' I_2'} = \frac{1}{2}(\delta^{I_1}_{I_1'} \delta^{I_2}_{I_2'} - \delta^{I_1}_{I_2'} \delta^{I_2}_{I_1'}) \tag{27}$$

have to be satisfied by definition, provided the wave functions $\{C_k\}$ are a complete set of antisymmetric functions in (I_1, I_2)-space.

In order to show the equivalence of both sides of (24), we study the inversion of (24) by multiplying

(24) with the corresponding duals and summing over $I_1 \dots I_{2n}$. Using here and in the following the summation convention for k, I, etc., this gives

$$R^{k_1}_{I_1 I_2} \dots R^{k_n}_{I_{2n-1} I_{2n}} \varphi(I_1 \dots I_{2n}|a) = \tilde{\varrho}(k_1 \dots k_n|a) \tag{28}$$

with

$$\tilde{\varrho}(k_1 \dots k_n|a) = S^{k_1 \dots k_n}_{k'_1 \dots k'_n} \varrho(k'_1 \dots k'_n|a), \tag{29}$$

where

$$S^{k_1 \dots k_n}_{k'_1 \dots k'_n} = \frac{1}{(2n)!} \sum_{I_1 \dots I_{2n}} R^{k_1}_{I_1 I_2} \dots R^{k_n}_{I_{2n-1} I_{2n}} \cdot C^{\{I_1 I_2}_{k'_1} \dots C^{I_{2n-1} I_{2n}\}}_{k'_n}. \tag{30}$$

The latter tensors have the projector properties

$$S^{k_1 \dots k_n}_{j_1 \dots j_n} S^{j_1 \dots j_n}_{l_1 \dots l_n} = S^{k_1 \dots k_n}_{l_1 \dots l_n}. \tag{31}$$

From this it has to be concluded that a bijective mapping for the functions appearing in (24) can only be achieved if (24) is replaced by, see [21, 23]:

$$\varphi_n(I_1 \dots I_{2n}|a) = \frac{1}{(2n)!} \tilde{\varrho}(k_1 \dots k_n|a) \cdot C^{\{I_1 I_2}_{k_1} \dots C^{I_{2n-1} I_{2n}\}}_{k_n}. \tag{32}$$

Then according to (31) formula (28) is indeed the inverse of (32).

In the next step we define the generating functional state for bosons by

$$|\mathcal{F}(b|a)\rangle = \sum_{n=1}^{\infty} \frac{1}{n!} \tilde{\varrho}(k_1 \dots k_n|a) b_{k_1} \dots b_{k_n} |0\rangle_B \tag{33}$$

with the functional creation operators b_k and the functional Fock vacuum $|0\rangle_B$, where with the duals ∂^b_k of b_k the commutation relations

$$[b_k, \partial^b_{k'}]_- = \delta_{kk'} \tag{34}$$

are postulated, while all other commutators vanish.

Furthermore, for later use we define the projector

$$\mathcal{P} = \sum_{n=1}^{\infty} \frac{1}{n!} b_{l_1} \dots b_{l_n} |0\rangle_B S^{l_1 \dots l_n}_{l'_1 \dots l'_n} {}_B\langle 0| \partial^b_{l'_n} \dots \partial^b_{l'_1}, \tag{35}$$

which leaves the boson state (33) invariant:

$$\mathcal{P}|\mathcal{F}(b|a)\rangle = |\mathcal{F}(b|a)\rangle. \tag{36}$$

Therefore, by the definition (16) of the fermion functional state, by the definition of the mapping relation (32) and the definition of the boson functional state (33) a bijective map between (16) and (33) is established.

It remains the task to transform the eigenvalue equation (18) for the fermion state (16) into a corresponding eigenvalue equation for the boson state (33).

The first step must be to replace the mapping definition in configuration space (32) by a corresponding mapping definition in functional space.

By generalizing relation (22) the following definition was introduced by Kerschner [23]:

$$\mathcal{T}(b, \partial^f) := \sum_{n=1}^{\infty} \frac{1}{n!} b_{k_1} \dots b_{k_n} |0\rangle_B \cdot R^{k_1}_{I_1 I_2} \dots R^{k_n}_{I_{2n-1} I_{2n}} {}_f\langle 0| \partial^f_{I_1} \dots \partial^f_{I_{2n}}, \tag{37}$$

which leads in analogy to (23) to the mapping definition

$$|\mathcal{F}(b|a)\rangle = \mathcal{T}(b, \partial^f) |\mathcal{F}(j|a)\rangle, \tag{38}$$

and owing to the general definition of duals (26), (27) the inverse relation of (38) reads

$$|\mathcal{F}(j|a)\rangle = \mathcal{S}(j, \partial^b) |\mathcal{F}(b|a)\rangle \tag{39}$$

with

$$\mathcal{S}(j, \partial^b) := \sum_{n=1}^{\infty} \frac{1}{(2n)!} j_{I_1} \dots j_{I_{2n}} |0\rangle_f \cdot C^{I_1 I_2}_{k_1} \dots C^{I_{2n-1} I_{2n}}_{k_n} {}_B\langle 0| \partial^b_{k_n} \dots \partial^b_{k_1}. \tag{40}$$

In contrast to nuclear physics the inverse operator of $\mathcal{T}$ is not its Hermitean conjugate, but (40).

For these operators the following relation can be derived, [23]:

$$\mathcal{T}(b, \partial^f) \mathcal{S}(j, \partial^b) = \mathcal{P} \tag{41}$$

and

$$\mathcal{S}(j, \partial^b) \mathcal{T}(b, \partial^f) = \mathbf{1}(j, \partial^f). \tag{42}$$

Using these relations, equation (18) can be mapped into equation

$$E^a_0 |\mathcal{F}(b|a)\rangle = \mathcal{H}_B(b, \partial^b) |\mathcal{F}(b|a)\rangle \tag{43}$$

for the boson functional states (33) with

$$\mathcal{H}_B(b,\partial^b) = \mathcal{T}(b,\partial^f)\mathcal{H}_F(j,\partial^f)\mathcal{S}(j,\partial^b). \quad (44)$$

To evaluate (44), the commutator between $\mathcal{T}$ and $\mathcal{H}_F$ has to be derived. This can be achieved by the combination of two special commutation relations, [23]:

$$\mathcal{T}(b,\partial^f)j_I = 2R^k_{IK}b_k\mathcal{T}(b,\partial^f)\partial^f_K \quad (45)$$

and

$$\mathcal{T}(b,\partial^f)\partial^f_I\partial^f_K = C^{KI}_k\partial^b_k\mathcal{T}(b,\partial^f) \quad (46)$$

and repeated application. Similar commutator relations were evaluated and applied in nuclear physics, [19], Eqs. (2,7), (2,8), etc.

For the case under consideration one obtains

$$\begin{aligned} E^a_0|\mathcal{F}(b|a)\rangle := \{&K^{kk'}b_k\partial^b_{k'} + Q^k(b_k + \Gamma^{ll'}_{kk'}b_lb_{l'}\partial_{k'}) + W^{kll'}_1 b_k\partial^b_l\partial^b_{l'} + W^{kk'}_2(b_k + \Gamma^{ll'}_{kk''}b_lb_{l'}\partial_{k''})\partial_{k'} \\ &+ W^{kk'l}_3(b_k + \Gamma^{l'l''}_{kk''}b_{l'}b_{l''}\partial_{k''})b_{k'}\partial_l + W^{k_1k_2}_4(b_{k_1} + \Gamma^{l_1l'_1}_{k_1k'_1}b_{l_1}b_{l'_1}\partial_{k'_1})(b_{k_2} + \Gamma^{l_2l'_2}_{k_2k'_2}b_{l_2}b_{l'_2}\partial_{k'_2})\}|\mathcal{F}(b|a)\rangle, \end{aligned} \quad (47)$$

where the various tensor coefficients are defined by the following relations

$$\begin{aligned} K^{kk'} &:= 2K_{I_1I_2}R^k_{I_1K}C^{I_2K}_{k'}, \quad Q^k = 2K_{I_1I}F_{II_2}R^k_{I_1I_2}, \\ W^{kll'}_1 &:= 2U_{I_1I_2I_3I_4}R^k_{I_1K}C^{I_4K}_lC^{I_2I_3}_{l'}, \quad W^{kk'}_2 := -6U_{I_1I_2I_3I_4}F_{I_4I}R^k_{I_1I}C^{I_2I_3}_{k'}, \\ W^{kk'l}_3 &:= 4U_{I_1I_2I_3I_4}(3F_{I_4I}F_{I_3I'} + \frac{1}{4}A_{I_4I}A_{I_3I'})R^k_{I_1I}R^{k'}_{I'K}C^{I_2K}_l, \\ W^{k_1k_2}_4 &:= -4U_{I_1I_2I_3I_4}(F_{I_4I}F_{I_3I'} + \frac{1}{4}A_{I_4I}A_{I_3I'})F_{I_2I''}R^{k_1}_{I_2I}R^{k_2}_{I'I''}, \\ \Gamma^{ll'}_{kk'} &:= -2C^{I_1I_2}_kR^l_{I_2I_3}C^{I_3I_4}_{k'}R^{l'}_{I_4I_1}. \end{aligned} \quad (48)$$

The effective boson functional (47) can be transformed into those equations in [21,22], which were derived earlier without using the formalism introduced above.

4. Mappings with Two-fermion and Three-fermion States

Phenomenological theories describe the interactions of bosons and fermions. If a partonic substructure of these particles is assumed, this leads to the necessity to derive effective dynamical laws for the simultaneous occurrence of composite bosons and composite fermions. In this respect, at present, the most prominent example is given by the quark structure of mesons and nucleons in nuclear physics. In the simplest case this leads to the problem of a simultaneous treatment of two-fermion (quark) and three-fermion (quark) states.

A similar situation arises in the parton model of elementary particles, where gauge bosons are assumed to be composed of two partons, while leptons and quarks are assumed to have a three parton structure.

In both cases the elementary quark or parton dynamics, respectively, can be described by appropriately regularized NJL-models like that defined by the Lagrangian (1), and thus from these models an effective dynamical law for the composite bosons and fermions has to be derived.

Concerning the literature of such more complicated mappings, in physical Fock space the two-fermion-one-fermion problem was discussed in detail by Meyer, [24]. The same problem, referred to the algebraic Schroedinger respresentation in functional space, was treated by Kerschner, [23]. In addition Kerschner, [23], developed a derivation of an effective theory for the two-fermion-three-fermion system, which is of interest with respect to our problem. Unfortunately, in the latter case the corresponding proof has to be rejected, because it does not properly take into account the boundary conditions and the effect of dressed particle states on the mapping. Thus a new line of approach is necessary to solve this problem.

In this section we first treat the formal derivation of the mapping procedure, while in the following sections the physical implications of this proceeding are to be studied, and corresponding modifications are to be introduced.

The elements of the map are the two-fermion and the three-fermion states. The former are defined by the boson states (25), (26) and (27), while for the latter states we postulate the general orthonormality relations between wave functions C_q and their duals R^q

$$C_q^{I_1 I_2 I_3} R^{q'}_{I_1 I_2 I_3} = \delta_q^{q'}, \tag{49}$$

and the completeness relations

$$\sum_q C_q^{I_1 I_2 I_3} R^q_{I_1' I_2' I_3'} = \frac{1}{3!} \delta^{I_1 I_2 I_3}_{\{I_1' I_2' I_3'\}_{as}}. \tag{50}$$

The map of the parton state (16) onto a corresponding functional state for composite bosons and fermions is achieved by a generalization of the transformation (32) to the case under consideration. The boundary conditions will be incorporated afterwards.

For brevity we replace the parton indices I_l simply by l , i. e., $R_{I_1 I_2}$ is replaced by $R_{(1,2)}$, apply the summation convention for repeated numbers, and define the transformation by

$$\begin{aligned}\varphi_n(I_1 \dots I_n|a) = \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \delta_{n,2m+3r} \tilde{\varrho}(k_1 \dots k_m, q_1 \dots q_r) C_{k_1}^{\{(1,2)} \dots C_{k_m}^{(2m-1,2m)} \\ \cdot\ C_{q_1}^{(2m-1,2m+2,2m+3)} \dots C_{q_r}^{(2m+3r-2,2m+3r-1,2m+3r)\}}\end{aligned} \tag{51}$$

with

$$\tilde{\varrho}(k_1 \dots k_m, q_1 \dots q_r) = S^{k_1 \dots k_m, q_1 \dots q_r}_{k_1' \dots k_{m'}', q_1' \dots q_{r'}'} \varrho(k_1' \dots k_{m'}', q_1' \dots q_{r'}') \tag{52}$$

and

$$\begin{aligned}S^{k_1 \dots k_m, q_1 \dots q_r}_{k_1' \dots k_{m'}', q_1' \dots q_{r'}'} := \frac{1}{n!} R^{k_1}_{(1,2)} \dots R^{k_m}_{(2m-1,2m)} R^{q_1}_{(2m+1,2m+2,2m+3)} \dots R^{q_r}_{(2m+3r-2,2m+3r-1,2m+3r)} \\ \cdot\ C_{k_1'}^{\{(1,2)} \dots C^{(2m'-1,2m')} C_{q_1'}^{(2m'+1,2m'+2,2m'+3)} \dots C_{q_{r'}'}^{(2m'+3r'-2,2m'+3r'-1,2m'+3r')\}}.\end{aligned} \tag{53}$$

In analogy to (31), the matrix (53) has the projector property.

Using the the functional formulation, in this case the state (16) is mapped into its image functional state

$$|\mathcal{F}(b,f;a)\rangle := \sum_{n=2}^{\infty} \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \delta_{n,2m+3r} \frac{1}{(m!)(r!)} \tilde{\varrho}(k_1 \dots k_m, q_1 \dots q_r|a) b_{k_1} \dots b_{k_m} f_{q_1} \dots f_{q_r} |0\rangle_{BF} \tag{54}$$

by application of the operator

$$\begin{aligned}\mathcal{T}(b,f,\partial^f) := \sum_{n=2}^{\infty} \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \delta_{n,2m+3r} \frac{1}{(m!)(r!)} b_{k_1} \dots b_{k_m} f_{q_1} \dots f_{q_r} |0\rangle_{BF} R^{k_1}_{(1,2)} \dots R^{k_m}_{(2m-1,2m)} \\ \cdot\ R^{q_1}_{(2m+1,2m+2,2m+3)} \dots R^{q_r}_{(2m+3r-2,2m+3r-1,2m+3r)} {}_f\langle 0|\partial^f_n \dots \partial^f_1,\end{aligned} \tag{55}$$

where the boson operators satisfy the commutation relations (37), and the fermion operators satisfy the anticommutation relations

$$[f_q, \partial^F_{q'}]_+ = \delta_q^{q'}, \tag{56}$$

while all other anticommutators vanish, and the definition $|0\rangle_{BF} := |0\rangle_B \otimes |0\rangle_F$ for the functional vacuum state is used.

It should be noted and it is remarkable that the value $n = 1$ is not included in equation (54) as well as in (55), because it is impossible to find within the composite particle set a composite particle combination for $n = 1$. As in the general case all *n*-numbers are required for a nondegenerate map, the ommision of the $n = 1$ term has to be justified in order to obtain a selfconsistent theory. After having completely derived the effective dynamics of two- and three-fermion states we will treat this problem in section 6 in detail. Here, we only indicate that the omission of the $n = 1$ term is closely connected with the requirement that the three parton states describing leptons and quarks have to be genuine three particle states and are not allowed to have projections into the one particle (parton) sector.

If one defines

$$\mathcal{S}(j,\partial^b,\partial^f) := \sum_{n=2}^{\infty}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\delta_{n,2m+3r}\frac{1}{n!}j_1\dots j_n|0\rangle_f C_{k_1}^{(1,2)}\dots C_{k_m}^{(2m-1,2m)} \cdot C_{q_1}^{(2m+1,2m+2,2m+3)}\dots C_{q_r}^{(2m+3r-2,2m+3r-1,2m+3r)}{}_{BF}\langle 0|\partial^f_{q_r}...\partial^f_{q_1}\partial^b_{k_m}...\partial^b_{k_1} \tag{57}$$

and

$$\mathcal{P} := \sum_{n=0}^{\infty}\sum_{m,m'=0}^{\infty}\sum_{r,r'=0}^{\infty}\frac{1}{(n!)(m!)}b_{k_1}\dots b_{k_m}f_{q_1}\dots f_{q_r}|0\rangle_{BF}S^{k_1\dots k_m,q_1\dots q_r}_{k'_1\dots k'_{m'},q'_1\dots q'_{r'}}{}_{BF}\langle 0|\partial^f_{q'_{r'}}\dots\partial^f_{q'_1}\partial^b_{k'_{m'}}\dots\partial^b_{k'_1}, \tag{58}$$

analogous relations to equations (36), (38), (39), (41) – (44) hold for these operators, while after rather lengthy rearrangements equations (45) and (46) are replaced by

$$\mathcal{T}j_K = 2b_k R^k_{IK}\mathcal{T}\partial_I + 3f_q R^q_{II'K}\mathcal{T}\partial_{I'}\partial_I, \tag{59}$$

$$\mathcal{T}\partial_{K'}\partial_K = C^k_{KK'}\partial^b_k\mathcal{T} \tag{60}$$

and

$$\mathcal{T}\partial_{I_3}\partial_{I_2}\partial_{I_1} = C^q_{I_1I_2I_3}\partial^f_q\mathcal{T}. \tag{61}$$

For brevity all details of these calculations were suppressed.

Thus, by means of these relations equation (18) can be mapped into the two-fermion-three-fermion sector.

Before doing this in detail, we rewrite the energy operator (19) in the form

$$\mathcal{H}_F(j,\partial) := \mathcal{H}^d_F(j,\partial) + \mathcal{Q}_F(j,\partial) \tag{62}$$

with

$$\begin{aligned}\mathcal{H}^d_F(j,\partial) &= j_{I_1}K_{I_1I_2}\partial_{I_2}\\ &+ W_{I_1I_2I_3I_4}[j_{I_1}\partial_{I_4}\partial_{I_3}\partial_{I_2} - 3F_{I_4K}j_{I_1}j_K\partial_{I_3}\partial_{I_2}]\end{aligned} \tag{63}$$

and

$$\begin{aligned}\mathcal{Q}_F &= W_{I_1I_2I_3I_4}\\ &\cdot\Big[(3F_{I_4K_1}F_{I_3K_2} + \frac{1}{4}A_{I_4K_1}A_{I_3K_2})j_{I_1}j_{K_1}j_{K_2}\partial_{I_2}\\ &- (F_{I_4K_1}F_{I_3K_2} + \frac{1}{4}A_{I_4K_1}A_{I_3K_2})\\ &\cdot F_{I_2K_3}j_{I_1}j_{K_1}j_{K_2}j_{K_3}\Big].\end{aligned} \tag{64}$$

Obviously $\mathcal{Q}_F$ has to be interpreted as a generalized quantization term, where the anticommutators $A_{II'}$, which govern and define the abstract algebra, are modified by the influence of the vacuum, i. e., the influence of the special representation which is under consideration. In addition, if this representation is mapped into the two-fermion-three-fermion sector, further deformations of the original operator quantization rules are to be expected.

Because we are primarily interested in the effective dynamical law of the two-fermion-three-fermion sector, we postpone the investigation of these modified quantization terms and concentrate on the transformation of the terms in (63) which decribe the system dynamics (under the influence of the vacuum!).

In particular one recognizes that under the influence of the vacuum the system develops two-body forces which are given by the last term in (63).

Transformation of $\mathcal{H}^d_F$ leads to the equation

$$\begin{aligned}E^a_0|\mathcal{F}(b,f|a)\rangle = \Big\{&(K_{kl}+M_{kl})b_k\partial^b_l\\ &+ (K_{qp}+M_{qp})f_q\partial^f_p + W_1^{kl_1l_2}b_k\partial^b_{l_2}\partial^b_{l_1}\\ &+ W_2^{qlp}f_q\partial^b_l\partial^f_p + W_3^{k_1k_2l_1l_2}b_{k_1}b_{k_2}\partial^b_{l_1}\partial^b_{l_2}\\ &+ W_4^{q_1q_2p_1p_2}f_{q_1}f_{q_2}\partial^f_{p_1}\partial^f_{p_2}\\ &+ W_5^{qpkl}b_k\partial^b_l f_q\partial^f_p\Big\}|\mathcal{F}(b,f|a)\rangle\end{aligned} \tag{65}$$

with

$$K_{kl} := 2R^k_{II_1}K_{I_1I_2}C^l_{I_2I},$$

$$
\begin{aligned}
M_{kl} &:= -6W_{I_1I_2I_3I_4}F_{I_4K}R^k_{KI_1}C^l_{I_2I_3},\\
K_{qp} &:= 3R^q_{II'I_1}K_{I_1I_2}C^p_{I_2II'},\\
M_{qp} &:= -9W_{I_1I_2I_3I_4}F_{I_4K}\\
&\quad\cdot(R^q_{IKI_1}C^p_{I_2I_3I} - R^q_{II'K}C^p_{I_2I_3I'}),\\
W_1^{kl_1l_2} &:= 2W_{I_1I_2I_3I_4}2R^k_{II_1}C^{l_1}_{I_4I}C^{l_2}_{I_2I_3},\\
W_2^{qlp} &:= 3W_{I_1I_2I_3I_4}R^q_{II'I_1}C^q_{I_4II'}C^l_{I_2I_3}, \qquad (66)\\
W_3^{k_1k_2l_1l_2} &:= 12W_{I_1I_2I_3I_4}F_{I_4K}R^{k_1}_{II_1}R^{k_2}_{I'K}C^{l_1}_{II'}C^{l_2}_{I_2I_3},\\
W_4^{q_1q_2p_1p_2} &:= -27W_{I_1I_2I_3I_4}F_{I_4K}R^{q_1}_{II'I_1}R^{q_2}_{K_1K_2K}\\
&\quad\cdot C^{p_1}_{I'K_1K_2}C^{q_2}_{I_2I_3I},\\
W_5^{qpkl} &:= -24W_{I_1I_2I_3I_4}F_{I_4K}\Big(R^q_{I'I''K}C^p_{I_2I_3I}R^k_{II_1}\\
&\quad\cdot C^l_{I'I''} - R^q_{II'I_1}C^p_{I_2I_3I}R^k_{K'K}C^l_{I'K'}\Big).
\end{aligned}
$$

For the evaluation of the coefficient functions (66) the wave functions of the corresponding particles are required. With respect to these wave functions and their calculation, detailed investigations were performed in several papers. The boson wave functions can be exactly calculated, see [13, 14]. The fermion functions are discussed in Section 6. In Sects. 5 and 6 the summation convention will again be applied.

5. Constraints by Dressed Particle States

Formally, the mapping calculations of Sect. 3 are exact, provided the necessary preconditions are observed. But even if all these conditions are satisfied, physically, the map is inadequate. This inadequacy stems from the fact that by its construction the mapping formalism of Sect. 4 does not incorporate the effects of dressed particle states.

The existence of dressed particle states induces additional correlations which must be incorporated into the dynamics in order to narrow down the set of possible solutions of Equation (65) to those solutions which are physically meaningfull. Therefore the formalism of Sect. 4 has to be supplemented by considering the influence of such dressed states on the effective dynamics.

Two decades ago a proposal was made, [25], to solve this problem. Based on the development of the formalism of dressed particle states, [26], [13], [14], this proposal can now be elaborated in more detail, and the result can be incorporated in (65). Concerning the dressed particle formalism, we refer to [14] and take over the formulas and results needed for our discussion from [14]. In particular, the dressed boson states are of interest with respect to the modification of the effective dynamics in (65), i. e., we concentrate on the treatment of these states in the following.

As an intermediate step we deviate from the method of Sect. 2 and 3 and introduce an explicit functional representation of the dressed boson states by the ansatz, see [14], Eq. (5.50)

$$
b_{2j,k} = \sum_{n=0}^{\infty} C^{I_1\ldots I_{2j+2n}}_{2j,k} j_{I_1}\ldots j_{I_{2j+2n}}, \quad j = 1,2,\ldots\infty. \qquad (67)
$$

The index *j* describes the number of fermion pairs which lead to irreducible boson states, while *n* characterizes the contribution of the polarization part resulting from the excitation of fermion pairs, etc., in the vacuum. Furthermore, by the definition of dressed boson states any term in the expansion (67) must belong to the same quantum numbers, as otherwise a unique description and characterization of dressed boson states would be impossible.

The inversion of (67) reads, see [14], Eq. (5.52),

$$
j_{I_1}\ldots j_{I_{2r}} = \sum_k \sum_{j=1}^{r} R^{2j,k}_{I_1\ldots I_{2r}} b_{2j,k}, \qquad (68)
$$

where the *R*-functions in (68) are the duals of the *C*-functions in (67), see [14]. For fixed *k* and *j*, the duals $R^{2j,k}$ are projectors, which from the set of possible boson generators $b_{2h,l}$ project out the generators representing the quantum numbers *k*, *j* and no other ones. Hence the duals $R^{2j,k}$ must be elements of the same irreducible representation to which $b_{2j,k}$ belongs.

For $r = 3$ one obtains from (68)

$$
j_{I_1}\ldots j_{I_6} = \sum_k \Big[R^{2,k}_{I_1\ldots I_6} b_{2,k} + R^{4,k}_{I_1\ldots I_6} b_{4,k} + R^{6,k}_{I_1\ldots I_6} b_{6,k}\Big]. \qquad (69)
$$

To resolve this formula, we represent the fermion states of Sect. 3 in the explicit form

$$
f_{3,l} := C^{I_1I_2I_3}_{3,l} j_{I_1} j_{I_2} j_{I_3} \qquad (70)
$$

with the inversion

$$
j_{I_1} j_{I_2} j_{I_3} = \sum_l R^{3,l}_{I_1I_2I_3} f_{3,l}. \qquad (71)
$$

Then, multiplying (69) by the antisymmetrized product $C^{3,l} \otimes C^{3,l'}$ of two fermion states, for the left hand side of (69) the following relation

$$\sum_{\lambda_1 \dots \lambda_6} \frac{1}{6!} (-1)^P C^{3,l}_{I_{\lambda_1} I_{\lambda_2} I_{\lambda_3}} C^{3,l'}_{I_{\lambda_4} I_{\lambda_5} I_{\lambda_6}} j_{I_1} \dots j_{I_6} = C^{3,l}_{I_1 I_2 I_3} j_{I_1} j_{I_2} j_{I_3} C^{3,l'}_{I_4 I_5 I_6} j_{I_4} j_{I_5} j_{I_6} = f_{3,l} f_{3,l'} \tag{72}$$

results, and thus (69) is transformed into

$$f_{3,l} f_{3,l'} = \sum_{\lambda_1 \dots \lambda_6} (-1)^P \frac{1}{6!} \sum_k C_{3,l}^{I_{\lambda_1} I_{\lambda_2} I_{\lambda_3}} C_{3,l'}^{I_{\lambda_4} I_{\lambda_5} I_{\lambda_6}} \cdot \Big[R^{2,k}_{I_1 \dots I_6} b_{2,k} + R^{4,k}_{I_1 \dots I_6} b_{4,k} + R^{6,k}_{I_1 \dots I_6} b_{6,k} \Big]. \tag{73}$$

If we are not interested in the coupling of the higher order boson states $b_{4,k}$ and $b_{6,k}$ to the fermionic current, we can suppress the corresponding terms in (73) and obtain from (73)

$$f_{3,l} f_{3,l'} = \sum_k C_{3,l}^{I_1 I_2 I_3} C_{3,l'}^{I_4 I_5 I_6} R^{2,k}_{I_1 I_2 I_3 I_4 I_5 I_6} b_{2,k} = \sum_k \tilde{R}^k_{ll'} b_{2,k}, \tag{74}$$

if the antisymmetry of $R^{2,k}$ in $I_1, \dots, I_6$ is observed.

In (74) $C^{3,l} \otimes C^{3,l'}$ acts as projector which then and then only gives a nonvanishing expression if the quantum numbers attributed to this projector coincide with those of $b_{2,k}$. For electroweak bosons $b_{2,k}$ and fermion functions $C^{3,l}$, $C^{3,l'}$, this is the case if the fermions are assumed to be leptons or quarks, and if in the corresponding projectors these functions are chosen in an appropriate group theoretical combination. Hence from these fermion state combinations, a nontrivial contribution to the polarization cloud of the electroweak bosons is expected.

Having derived relation (74), we consider it as a constraint which relates (composite) fermions to bosons, and we return to the exact mapping theorems:

We disregard the functional operator relation (74) and consider the set $\{\tilde{R}^k_{ll'}\}$ as the starting point for a bosonization in the functional *F*-space of the effective theory (65).

If the set $\{\tilde{R}^k_{ll'}\}$ is assumed to be complete, one can construct the corresponding set of dual states $\{\tilde{C}^k_{ll'}\}$ and can thus apply the mapping procedure to Equation (65).

In so doing, we consider the boson operators in (65) as spectators and transform the *F*-space into an *F*-space and a boson space with generating sources $\tilde{b}$. After having performed this map, we identify the $\tilde{b}$-generators with the original (spectator) *b*-generators to achieve selfconsistency with the constraint (74). This step is justified as long as phenomenologically the electroweak bosons are only characterized by their quantum numbers. In the effectivce theory it is then impossible to discriminate $b_{2,k}$ from $\tilde{b}_{2,k}$.

The exact treatment of this (2,1)-map runs along the same lines as that of Sect. 2 and was performed by Kerschner in [23]. To avoid lengthy deductions we refer directly to the corresponding formulas of [23].

To apply the (2,1)-map to the effective energy operator of (65) we rearrange it into the form

$$\mathcal{H}_{eff}(b, f, \partial^b, \partial^f) := \mathcal{H}_1(b, f, \partial^b, \partial^f) + \mathcal{H}_2(b, f, \partial^b, \partial^f) \tag{75}$$

with

$$\begin{aligned} \mathcal{H}_1 &:= \tilde{K}_{qp} f_q \partial^f_p + W_4^{q_1 q_2 p_1 p_2} f_{q_1} f_{q_2} \partial^f_{p_1} \partial^f_{p_2} \\ \mathcal{H}_2 &:= (K^b_{kl} + M^b_{kl}) b_k \partial^b_l + W_1^{k l_1 l_2} b_k \partial^b_{l_1} \partial^b_{l_2} \\ &\quad + W_3^{k_1 k_2 l_1 l_2} b_{k_1} b_{k_2} \partial^b_{l_1} \partial^b_{l_2}, \end{aligned} \tag{76}$$

and

$$\tilde{K}_{qp} := (K^f_{qp} + M^f_{qp} + W_2^{qlp} \partial^b_l + W_5^{qpkl} b_k \partial^b_l). \tag{77}$$

Denoting the corresponding mapping operators by $\tilde{\mathcal{T}}$ and $\tilde{\mathcal{S}}$, from [23], Eq. (3.305) it follows

$$\tilde{\mathcal{T}} f_q \partial^f_p \tilde{\mathcal{S}} = [2 \tilde{R}^{k_1}_{pr} \tilde{C}^{qr}_{k_2} \tilde{b}_{k_1} \tilde{\partial}^b_{k_2} + f_q \partial^f_p] \tilde{\mathcal{P}}. \tag{78}$$

Furthermore, from [23], Eq. (3.303) one obtains

$$\begin{aligned} \tilde{\mathcal{T}} f_{q_1} f_{q_2} \partial^f_{p_1} \partial^f_{p_2} \tilde{\mathcal{S}} = \Big[& 2 \tilde{R}^k_{q_1 q_2} (\tilde{b}_k + \tilde{\Gamma}^{ll'}_{kk'} \tilde{b}_l \tilde{b}_{l'} \tilde{\partial}^b_{k'}) \\ & - 2 (\tilde{R}^k_{q_1 r_2} \delta^{r_1}_{q_2} - \tilde{R}^k_{q_2 r_2} \delta^{r_1}_{q_1}) \tilde{b}_k f_{r_1} \partial^f_{r_2} \\ & + f_{q_1} f_{q_2} \Big] \partial^f_{p_1} \partial^f_{p_2} \tilde{\mathcal{P}}. \end{aligned} \tag{79}$$

Having derived these relations, one can transform (65) into the $\tilde{b}$-f space by applying the rules of Sect. 2 and 3. This gives with

$$\tilde{\mathcal{T}} |\mathcal{F}(b, f|a)\rangle = |\mathcal{F}(\tilde{b}, b, f|a)\rangle \tag{80}$$

for equation (65) the expression

$$E_0^a|\mathcal{F}(\tilde{b},b,f|a)\rangle\!\rangle = \tilde{\mathcal{T}}\mathcal{H}_{eff}(b,f,\partial^b,\partial^f)\tilde{\mathcal{S}}\tilde{\mathcal{T}}|\mathcal{F}(b,f|a)\rangle\!\rangle$$
$$= \tilde{\mathcal{H}}(\tilde{b},b,f,\tilde{\partial}^b,\partial^b,\partial^f)|\mathcal{F}(\tilde{b},b,f|a)\rangle, \tag{81}$$

because (36) holds likewise for the projector $\tilde{\mathcal{P}}$.

Introducing the constraint $\tilde{b} \equiv b$ means identifying the $\tilde{b}$ generators and their duals $\tilde{\partial}^b$ with the spectator generators b and their duals ∂^b.

Under these constraints one obtains from (81)

$$\tilde{\mathcal{H}} = \mathcal{H}_f(f,\partial^f) + \mathcal{H}_b(b,\partial^b) + \mathcal{H}_{bf}(b,f,\partial^b,\partial^f) \tag{82}$$

with

$$\mathcal{H}_f := (K^f_{qp} + M^f_{qp})f_q\partial^f_p + W_4^{q_1q_2p_1p_2} f_{q_1} f_{q_2}\partial^f_{p_1}\partial^f_{p_2}, \tag{83}$$

$$\mathcal{H}_b := [(K^b_{kl} + M^b_{kl}) + (K^f_{qp} + M^f_{qp})2\tilde{R}^k_{qr}\tilde{C}^{pr}_l]b_k\partial^b_l$$
$$+ [W_1^{kl_1l_2} + 2W_2^{ql_1p}\tilde{R}^k_{qr}\tilde{C}^{pr}_{l_2}]b_k\partial^b_{l_2}\partial^b_{l_1} \tag{84}$$
$$+ [W_3^{k_1k_2l_1l_2} + W_5^{qpk_1l_1}2\tilde{R}^{k_2}_{qr}\tilde{C}^{pr}_{l_2}]b_{k_1}b_{k_2}\partial^b_{l_1}\partial^b_{l_2}$$

and

$$\mathcal{H}_{bf} := W_2^{qlp}\partial^b_l f_q\partial^f_p + W_4^{q_1q_2p_1p_2}2\tilde{R}^k_{q_1q_2}b_k\partial^f_{p_1}\partial^f_{p_2}$$
$$+ W_5^{qpkl}b_k\partial^b_l f_q\partial^f_p + W_4^{q_1q_2p_1p_2} \tag{85}$$
$$\cdot[\tilde{\Gamma}^{ll'}_{kk'}b_lb_{l'}\partial^b_{k'} + (\tilde{R}^k_{q_1r_2}\delta^{r_1}_{q_2} - \tilde{R}^k_{q_2r_2}\delta^{r_1}_{q_1})b_k f_{r_1}\partial^f_{r_2}]\partial^f_{p_1}\partial^f_{p_2}.$$

For a complete identification of electroweak gauge bosons, leptons and quarks it would be necessary to evaluate their effective dynamics in detail, which hypothetically is defined by the functional energy operator (82). This would exceed the scope of this paper. It is, however, possible to discriminate analytically lepton and quark states without explicit evaluation of (82), as will be demonstrated in the following section.

Although in this paper the identification problem of composite particles is not explicitly treated by evaluating (82), it should be emphasized: for the supposition that (82) is a meaningful description of the effective dynamics of these bosons and fermions it exists a strong evidence. If exchange forces are partly neglected an approximative expression for the effective dynamics can be formulated and derived by representing boson and fermion states directly in the form (67) and (70) instead of using the exact mapping formula (51). For this case several explicit evaluations were performed which allowed to identify bosons and fermions with the elementary particles of the standard model. For a review see [14].

6. Boundary Conditions for Fermion States

From (66) it follows that for the application and evaluation of the mapping theorem the explicit knowledge of the wave functions of the various particles is required. In this section we concentrate on the role of fermion states within this mapping formalism. In particular we will demonstrate why the omission of the $n = 1$ term in (54) and (55) can be justified and leads to a discrimination of leptons and quarks.

Starting with the Lagrangian density (1), by means of the field theoretic formalism generalized de Broglie-Bargmann-Wigner (GBBW)-equations for three-parton states can be derived, the solutions of which are assumed to represent lepton and quark states. The Lagrangian (1) and the corresponding GBBW-equations are invariant under the global $SU(2) \otimes U(1)$ superspin-isospin symmetry group, and the solutions of the GBBW-equations are to be representations of the corresponding permutation group, i.e., for three-parton states under the group S_3.

With respect to these groups as well as the Lorentz group, a group theoretical analysis was performed in [27], the result of which leads to the table

	χ^1	χ^2	χ^3	χ^4	χ^5	χ^6	χ^7	χ^8
t	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$	$1/2$
t_3	$1/2$	$-1/2$	$1/2$	$-1/2$	$1/2$	$-1/2$	$1/2$	$-1/2$
f	1	1	-1	-1	$1/3$	$1/3$	$-1/3$	$-1/3$
q	1	0	0	-1	$2/3$	$-1/3$	$1/3$	$-2/3$
	$\bar{e}$	$\bar{\nu}$	ν	e	u	d	$\bar{d}$	$\bar{u}$

for the classification of leptons and quarks by their internal superspin-isospin wave functions χ.

In this table the isospin quantum numbers t and t_3 are to be identified with the quantum numbers of the unbroken electroweak SU(2) gauge group, and no distinction is made between lefthanded and righthanded particles. The latter difference comes about by parity symmetry breaking and is not the topic of this investigation. The fermion number f in this scheme is connected with the weak hypercharge by $f = 2y$. Using this relation, the values of the table are in complete correspondence to the phenomenological values of the lefthanded particles, see for instance [28], Table 22.2.

Analogous values characterize the lepton and quark states of the higher generations, where the occurence of the latter states depends on the calculation of the mass spectrum to be done in the last step. In a preliminary way this was shown in [29] by calculations with the corresponding energy equation in the ultralocal limit. Furthermore, by means of the Clebsch-Gordan coefficients, isospin representations for $t = 3/2$ can be derived. After electroweak isospin symmetry breaking it is assumed that the corresponding masses are extremely large and are thus at present unobservable, i. e., for getting a more realistic information about the parton structure of leptons and quarks, electroweak symmetry breaking has to be taken into account.

Nevertheless, even if this symmetry breaking is included in the calculation of eigenstates of GBBW-equations, no explanation of the difference between lepton and quark states can be given within this formalism.

However, it will be demonstrated that after electroweak symmetry breaking an explanation of the difference between leptons and quarks can be given by observing the $n = 1$ term boundary condition for the mapping on to an effective theory.

To evaluate this condition it is necessary to change from charge conjugated spinors to *G*-conjugated spinors, as the group theoretical analysis in [27] is referred to these quantities.

This transformation reads

$$\psi_{\kappa\alpha i}(x) = G_{\kappa\kappa'}\tilde{\psi}_{\kappa'\alpha i}(x), \tag{86}$$

where $\tilde{\psi}$ is defined by the set (ψ, ψ^G) and *G* is defined by

$$G := \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -i\sigma_2 \end{pmatrix}. \tag{87}$$

With (86) and (87) the three-parton states $\varphi_{I_1 I_2 I_3}$ or $\hat{\varphi}$, respectively, are transformed into

$$\begin{aligned} &\hat{\varphi}^{\kappa_1\kappa_2\kappa_3}_{\alpha_1\alpha_2\alpha_3}(x_1, x_2, x_3) = \\ &\qquad G_{\kappa_1\kappa_1'}G_{\kappa_2\kappa_2'}G_{\kappa_3\kappa_3'}\hat{\chi}^{\kappa_1'\kappa_2'\kappa_3'}_{\alpha_1\alpha_2\alpha_3}(x_1, x_2, x_3), \end{aligned} \tag{88}$$

and application to the superspin-isospin group generators leads to the transformation laws

$$G^{k'} = G^{-1}G^kG = \frac{1}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \frac{1}{2}\mathbf{1} \otimes \sigma^k \tag{89}$$

and

$$F' = G^{-1}FG = \frac{1}{3}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \equiv \frac{1}{3}\sigma^3 \otimes \mathbf{1}. \tag{90}$$

According to Sect. 4, the two-fermion-three-fermion map does not apply to the $n = 1$ sector of the original functional equation (18), (19). Hence this map is then and then only compatible with this equation if the $n = 1$ sector vanishes identically. This can be considered as a boundary condition imposed on the map.

To simplify the notation we use the covariant boundary condition for $n = 1$, which reads

$$(D^{\mu}_{II'}\partial_\mu - m_{II'})\varphi_{I'} = W_{II_1I_2I_3}\varphi_{I_1I_2I_3}. \tag{91}$$

This condition is completely equivalent to the corresponding condition in the algebraic Schroedinger representation (18), (19) and thus can replace the latter.

If condition (91) is transformed into the G-conjugated representation, this yields

$$\begin{aligned} &(D^{\mu}_{\alpha\alpha'}\partial_\mu - m_i\delta_{\alpha\alpha'})\delta_{ii'}\delta_{\Lambda\Lambda'}\delta_{AA'}\chi^{(\Lambda'A')}_{\alpha'}(x)_{i'} = \\ &\sum_h g\lambda_i \Big[v^h_{\alpha\alpha_1}\delta_{\Lambda\Lambda_2'}\delta_{AA_1'}(v^hC)_{\alpha_2\alpha_3}c_{\Lambda_2'\Lambda_3'}c_{A_2'A_3'} \\ &\quad + v^h_{\alpha\alpha_2}\delta_{\Lambda\Lambda_2'}\delta_{AA_2'}(v^hC)_{\alpha_3\alpha_1}c_{\Lambda_3'\Lambda_1'}c_{A_3'A_1'} \\ &\quad + v^h_{\alpha\alpha_3}\delta_{\Lambda\Lambda_3'}\delta_{AA_3'}(v^hC)_{\alpha_1\alpha_2}c_{\Lambda_1'\Lambda_2'}C_{A_1'A_2'}\Big] \\ &\qquad \cdot \hat{\chi}^{(\Lambda_1'A_1')(\Lambda_2'A_2')(\Lambda_3'A_3')}_{\alpha_1\alpha_2\alpha_3}(x, x, x). \end{aligned} \tag{92}$$

One can interprete (91) or (92), respectively, in the following way:

Let $\varphi^{(3)}$ be a nontrivial three-parton state amplitude which yields after substitution in (92) a nonvanishing right hand side. Then, if $\varphi^{(3)}$ describes a physical state, due to the field theoretic construction of (92) a wave function $\varphi^{(1)}$ has to exist with the same quantum numbers as $\varphi^{(3)}$. Hence $\varphi^{(3)}$ cannot be the most elementary description of this state. Therefore if leptons and quarks are to be genuine three-parton states, their wave functions must decouple from $\varphi^{(1)}$, i.e., substituted in (92) for these states the right hand side of (92) must vanish, which automatically includes the vanishing of $\varphi^{(1)}$.

Without worrying about the mechanism of the symmetry breaking one can accept its existence and ask for the group theoretical consequences with regard to the construction of the set of states under consideration. After the electroweak symmetry breaking has taken place only one algebraic constraint remains in the superspinor-isospinor space, namely the eigenvalue condition of the charge operator, see [28], p. 302. I.e., in the state space only the charge operator has to be diagonalized. This gives the condition

$$Q'_{I_1 I}\chi_{I I_2 I_3} + Q'_{I_2 I}\chi_{I_1 I I_3} + Q'_{I_3 I}\chi_{I_1 I_2 I} = q\chi_{I_1 I_2 I_3}, \tag{93}$$

where the solutions of this condition can be constructed by direct products of one particle states.

In the one-parton sector the eigenvalue condition reads

$$Q'_{\kappa_1\kappa'}\chi_{\kappa'\alpha}(x)_i = q\chi_{\kappa\alpha}(x)_i \tag{94}$$

with

$$Q' = G'_3 + \frac{1}{2}F' \equiv \frac{1}{2}\delta_{\Lambda_1\Lambda_2}\sigma^3_{A_1A_2} + \frac{1}{6}\sigma^3_{\Lambda_1\Lambda_2}\delta_{A_1A_2}. \tag{95}$$

The eigenvectors of the charge operator are derived by means of the ansatz $\chi_{\kappa\alpha}(x)_i = \zeta^l_\kappa\chi_\alpha(x,l)_i$ with $\zeta^l_\kappa = \delta_{\kappa,l}$, $l = 1,\dots,4$. For the three-parton case the eigenvectors are generated by the direct products $\zeta^{l_1}\otimes\zeta^{l_2}\otimes\zeta^{l_3}$ with $l_1,l_2,l_3 = 1,\dots,4$.

For unbroken symmetry, in general such direct products cannot be considered as appropriate eigensolutions, because the complete wave functions $\varphi^{(3)}$ or $\chi^{(3)}$, respectively, have to be antisymmetric. But with the loss of the full symmetry by symmetry breaking, the permutation symmetry is lost, too, see [30], and thus single direct products of one-parton functions can be considered as representatives of the various states. If each of these direct products is an eigenvector of (93), then there exists a total of 64 states, which, however, are highly degenerate in general.

To illustrate this we give the representatives of the set of degenerate states for the leptons, where the eigenfunctions Θ^j of superspin-isospin are simply given by direct products in the form

$$\Theta^j_{\Lambda_1A_1,\Lambda_2A_2,\Lambda_3A_3} := \delta_{A_1,\alpha^j_1}\delta_{A_2,\alpha^j_2}\delta_{A_3,\alpha^j_3} \cdot \delta_{\Lambda_1,\beta^j_1}\delta_{\Lambda_2,\beta^j_2}\delta_{\Lambda_3,\beta^j_3}, \tag{96}$$

which with α^j_i and β^j_i as the superspin-isospin quantum numbers of the single particle states leads to the table above.

Translated into the language of the original states, see [27], after symmetry breaking the leptonic states

Θ	α_1	α_2	α_3	β_1	β_2	β_3	q	f	
Θ^1	2	1	1	1	1	1	1	1	$\bar{e}$
Θ^2	2	2	1	1	1	1	0	1	$\bar{\nu}$
Θ^3	2	1	1	2	2	2	0	-1	ν
Θ^4	2	2	1	2	2	2	-1	-1	e

can be written in the same form with respect to the superspinor part. Furthermore, the symmetry breaking has no effect on (92). By substitution of these states into (92), the right hand side of this equation vanishes. Therefore in both cases the leptonic states decouple from the one-parton sector, remain irreducible states and the boundary condition for the map is satisfied because this implies that φ^1 vanishes too.

This situation is drastically changed for quark states. We consider the superspinor-isospinor representatives of the set of quark states given by the table

Θ	α_1	α_2	α_3	β_1	β_2	β_3	q	f	
Θ^5	2	1	1	1	1	2	$2/3$	$1/3$	u
Θ^6	2	2	1	1	1	2	$-1/3$	$1/3$	d
Θ^7	2	1	1	1	2	2	$1/3$	$-1/3$	$\bar{d}$
Θ^8	2	2	1	1	2	2	$-2/3$	$-1/3$	$\bar{u}$

which is referred to the representation of Θ^j by one particle superspinor-isospinor states (96).

After symmetry breaking, the superspinor-isospinor parts for quarks are expressed by such direct products, and substituting these states of the table into (92) one easily verifies that in this case the right hand side of (92) does not vanish. Therefore after symmetry breaking the quark states have lost their irreducibility.

There is only one way to restore this irreducibility: The local amplitude $\chi(x,x,x)$ has to vanish.

In order to achieve this, a group theoretical argument is desirable. For instance it would be sufficient if the quark states had nonvanishing orbital angular momenta in their groundstates. Then the difference between leptons and quarks would be created if the leptons had zero orbital momentum which is allowed only for the leptonic ground states, see above.

The decision whether this group theoretical argument works, obviously depends on the group theoretical properties of the three-parton wave function $C_q^{I_1I_2I_3}$ of Section 4. These wave functions cannot be chosen arbitrarily. On the contrary: they have to be calculated in accordance with the general field theoretic formalism of the model under consideration. This is guaranteed if the three-parton states are solutions of the corresponding GBBW-equations, see above.

About the solutions of these equations exact statements can be made: by a slight generalization of the results in [31], it can be shown that in the rest system the angular momentum operators $\mathbf{J}^2$ and J_3 commute with the operators defining the GBBW-equation, and by a slight modification of the results of [32], Sect. 5, withouut symmetry breaking the intrinsic partity operator $\mathcal{P}$ commutes with the GBBW-operators, too.

The simultaneous existence of the associated quantum numbers leads to the conclusion that in this case the orbital angular momentum quantum numbers must have definite values, too, see [33], Sect. 6.1 for a simple example. Owing to the relativistic invariance of the GBBW-equations and the angular momentum conditions these quantum numbers can be transferred to arbitrary **k**-vectors, so the complete set of three-parton states can be classified in this way.

Hence with respect to the three-parton solutions of GBBW-equations one can indeed verify the proposed grooup theoretical argument, as in any case the solutions without symmetry breaking are basic for further calculations. More details will be elaborated in [27].

The latter facts provide the basic for an approximate construction of quark states, proposed and elaborated in [34] and lead in consequence to a quantum mechanical explanation of color, [34]. This explanation is substantiated by group theoretical considerations, see [35], Sect. 11.1, where it is demonstrated that in quantum mechanics a representation space of the invariance group G with dimension n is simultaneously a representation space of $SU(n)$. For $n = 3$ in the case $O(3)$ this leads to a representation of $SU(3)$.

With respect to the interpretation of the effective dynamics this transmutation was already quantitatively studied in [34], see also [14], but a more detailed elaboration should be performed, including the results of recent progress. This will be done else where.

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